

Math 201, Fall 2010–11  
Calculus and Analytic Geometry III, sections 1–4  
Quiz 2, November 27 — Duration: 60 minutes

**GRADES:**

1 (/12)	2 (/14)	3 (/12)	4 (/14)	5 (/12)	6 (/14)	TOTAL/78

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**YOUR NAME:**

Questions &  
Correction

**YOUR AUB ID#:**

**PLEASE CIRCLE YOUR SECTION:**

Section 1	Section 2	Section 3	Section 4
Lecture MWF 3	Lecture MWF 3	Lecture MWF 3	Lecture MWF 3
Professor Makdisi	Professor Makdisi	Professor Makdisi	Professor Makdisi
Recitation F 11	Recitation F 5	Recitation F 4	Recitation F 10
Ms. Mroue	Ms. Mroue	Ms. Mroue	Ms. Mroue

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**INSTRUCTIONS:**

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit.
3. Each problem on this exam counts for 12 or 14 points. THE TOTAL NUMBER OF POINTS ON THE EXAM IS 78.
4. You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
5. Closed book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

**GOOD LUCK!**

An overview of the exam problems.  
**THE TOTAL NUMBER OF POINTS ON THIS EXAM IS 78.**

Take a minute to look at all the questions, THEN  
 solve each problem on its corresponding page INSIDE the booklet.

1. (12 pts total) Two UNRELATED questions about polar coordinates.
  - a) (2 pts) Given the curve  $C = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 3x + 4y\}$ . Write the equation of  $C$  in polar coordinates. (Cultural note:  $C$  is a circle.)
  - b) (10 pts) Let  $C'$  be the curve in polar coordinates whose equation is  $r = 1 + \sqrt{2} \cos \theta$ . Make a table of values, and draw the curve  $C'$ . **Make sure to label:** (i) the intercepts on both  $x$ - and  $y$ -axes, (ii) the tangent lines at the origin.

2. (14 pts total) Consider the moving point in space given by the position vector

$$P(t) = \left( t^2 + \frac{t^3}{3}, \ t^2 - \frac{t^3}{3}, \ t^2 \right).$$

- a) (8 pts) Find the arclength for the parametrized curve  $P(t)$  between  $t = 0$  and  $t = 1$ . Hint:  $\sqrt{at^4 + bt^2} = t\sqrt{at^2 + b}$ .
- b) (4 pts) Give an **approximation** for the displacement  $\Delta \vec{r}$  and the distance  $\Delta s$  traveled by the point  $P(t)$  in the short time interval between  $t = 1$  and  $t = 1.01$ .
- c) (2 pts) Is your approximation for  $\Delta s$  from part (b) larger or smaller than the true distance traveled between  $t = 1$  and  $t = 1.01$ ? **Explain why.**

3. (12 pts total) Given the function  $f(x, y, z) = x\sqrt{1 + y^2 + 2z^2}$ .

- a) (2 pts) Find the gradient  $\vec{\nabla}f$ .
- b) (6 pts) A point moves in  $\mathbf{R}^3$  with position vector  $\vec{r}(t)$ . We know that the position and velocity of the point at time  $t = 0$  are  $\vec{r}(t) \Big|_{t=0} = (1, 1, 1)$  and  $\frac{d\vec{r}}{dt} \Big|_{t=0} = (3, 4, 5)$ . Your job is to find  $\frac{d}{dt} f(\vec{r}(t)) \Big|_{t=0}$ .
- c) (4 pts) Starting from the point  $P_0 = (1, 1, 1)$ , in which direction does  $f$  decrease most rapidly? Your answer should be a unit vector.

4. (14 pts total) Consider the function  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  and its level set  $S$ , given by

$$f(x, y, z) = x^2 + 4z^2, \quad S = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + 4z^2 = 8\}.$$

- a) (3 pts) Sketch the surface  $S$ .
  - b) (6 pts) Find the equation of the tangent plane to  $S$  at the point  $P_0(2, -1, 1)$ .
  - c) (5 pts) Starting from the point  $P_0$  above, we move 0.1 units in the direction of the vector  $\vec{v} = (3, 4, 0)$ . Approximately how much is the resulting change in  $f$ ?
5. (12 pts) Find the maximum and minimum values of the function  $f(x, y) = xy$  on the elliptical disk  $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + 2y^2 \leq 1\}$ .

6. (14 pts total) Suggestion for both parts: use Taylor series and  $O(\cdot)$  notation.

- a) (7 pts) Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{\ln(1 + xy)}{x^2 + y^2}$  does NOT exist.
- b) (7 pts) Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{\ln(1 + x^2y^2)}{x^2 + y^2}$  DOES exist.

1. (12 pts total) Two UNRELATED questions about polar coordinates.

a) (2 pts) Given the curve  $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 3x + 4y\}$ . Write the equation of  $C$  in polar coordinates. (Cultural note:  $C$  is a circle.)

b) (10 pts) Let  $C'$  be the curve in polar coordinates whose equation is  $r = 1 + \sqrt{2} \cos \theta$ . Make a table of values, and draw the curve  $C'$ . Make sure to label: (i) the intercepts on both  $x$ - and  $y$ -axes, (ii) the tangent lines at the origin.

$$\begin{aligned} a) \quad x &= r \cos \theta \\ y &= r \sin \theta \\ x^2 + y^2 &= r^2 \end{aligned}$$

$$r^2 = 3r \cos \theta + 4r \sin \theta$$

can cancel  $r$  in this case to get

$$r = 3 \cos \theta + 4 \sin \theta$$

$\theta$	$\cos \theta$	$r = 1 + \sqrt{2} \cos \theta$
0	1	$1 + \sqrt{2} \approx 2.4$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	2
$\frac{\pi}{2} = \frac{2\pi}{4}$	0	1
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	0
$\pi = \frac{4\pi}{4}$	-1	$1 - \sqrt{2} \approx -0.4$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	0
$\frac{3\pi}{2} = \frac{6\pi}{4}$	0	1
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	2
$2\pi = \frac{8\pi}{4}$	1	$1 + \sqrt{2} \approx 2.4$

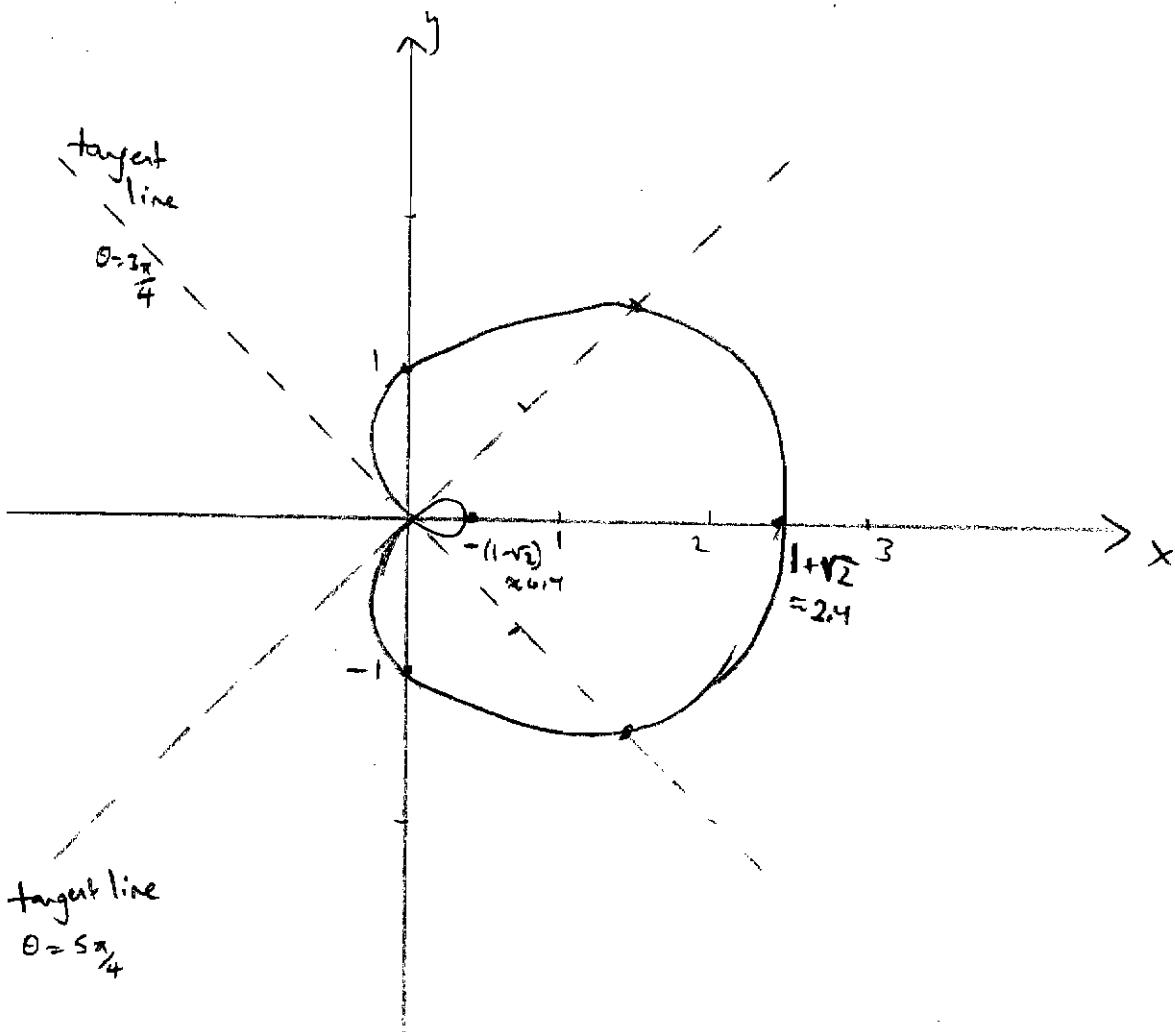
intercepts on the

$x$ -axis:  $\theta = 0, r = 1 + \sqrt{2}$   
 $\theta = \pi, r = 1 - \sqrt{2}$

$y$ -axis:  $\theta = \frac{\pi}{2}, r = 1$   
 $\theta = \frac{3\pi}{2}, r = 1$

when  $r = 0$   
 there are  
 tangent lines  
 at the origin  
 with  $\theta = \frac{3\pi}{4}$ ,  
 $\theta = \frac{5\pi}{4}$

corresponding to  
 $\theta = -\frac{1}{\sqrt{2}}$



2. (14 pts total) Consider the moving point in space given by the position vector

$$P(t) = \left( t^2 + \frac{t^3}{3}, t^2 - \frac{t^3}{3}, t^2 \right).$$

a) (8 pts) Find the arclength for the parametrized curve  $P(t)$  between  $t = 0$  and  $t = 1$ .

Hint:  $\sqrt{at^4 + bt^2} = t\sqrt{at^2 + b}$ .

b) (4 pts) Give an approximation for the displacement  $\Delta \vec{r}$  and the distance  $\Delta s$  traveled by the point  $P(t)$  in the short time interval between  $t = 1$  and  $t = 1.01$ .

c) (2 pts) Is your approximation for  $\Delta s$  from part (b) larger or smaller than the true distance traveled between  $t = 1$  and  $t = 1.01$ ? Explain why.

a)  $\vec{r} = \left( t^2 + \frac{t^3}{3}, t^2 - \frac{t^3}{3}, t^2 \right), \vec{v} = \frac{d\vec{r}}{dt} = (2t+t^2, 2t-t^2, 2t)$ ,

$$\text{speed} = \sqrt{(2t+t^2)^2 + (2t-t^2)^2 + (2t)^2} = \sqrt{4t^2 + 4t^3 + t^4 + 4t^2 - 4t^3 + t^4 + 4t^2} = \sqrt{2t^4 + 12t^2}$$

$$\begin{aligned} \text{arclength} &= \int_{t=0}^1 \sqrt{2t^4 + 12t^2} dt = \int_{t=0}^1 \sqrt{2t^2 + 12} \cdot t dt = \frac{1}{4} \int_{t=0}^1 \sqrt{2t^2 + 12} \cdot d(2t^2 + 12) \\ &= \left[ \frac{1}{4} \left( \frac{(2t^2 + 12)^{3/2}}{3/2} \right) \right]_{t=0}^1 = \left[ \frac{14^{3/2}}{6} - \frac{12^{3/2}}{6} \right] (\approx 1.8) \end{aligned}$$

b)  $t_0 = 1, \Delta t = 0.01, \Delta \vec{r} \approx \vec{v}|_{t_0} \Delta t = (2t+t^2, 2t-t^2, 2t)|_{t=1} \Delta t$   
 $= (3, 1, 2)(0.01) = (0.03, 0.01, 0.02)$

$$\Delta s = |\Delta \vec{r}| \approx \sqrt{(\vec{v}|_{t_0}) \Delta t} = \sqrt{(3, 1, 2)(0.01)} = \sqrt{14} / 100 (\approx 0.037)$$

c) As  $t$  increases from 1 to 1.01, the speed  $|\vec{v}| = \sqrt{2t^4 + 12t^2}$  increases from its initial value  $\sqrt{14}$  at  $t=1$  to something slightly larger.

Thus over the 0.01 seconds, the average speed is slightly larger than  $\sqrt{14}$ .

& the true distance traveled is slightly larger than  $\sqrt{14} \Delta t = \frac{\sqrt{14}}{100}$ .

3. (12 pts total) Given the function  $f(x, y, z) = x\sqrt{1+y^2+2z^2}$ .

a) (2 pts) Find the gradient  $\vec{\nabla}f$ .

b) (6 pts) A point moves in  $\mathbf{R}^3$  with position vector  $\vec{r}(t)$ . We know that the position and velocity of the point at time  $t=0$  are  $\vec{r}(t)|_{t=0} = (1, 1, 1)$  and  $\frac{d\vec{r}}{dt}|_{t=0} = (3, 4, 5)$ . Your job is to find  $\frac{d}{dt}f(\vec{r}(t))|_{t=0}$ .

c) (4 pts) Starting from the point  $P_0 = (1, 1, 1)$ , in which direction does  $f$  decrease most rapidly? Your answer should be a unit vector.

$$\text{a)} \quad \vec{\nabla}f = (f_x, f_y, f_z) = \left( \sqrt{1+y^2+2z^2}, \frac{x \cdot 2y}{2\sqrt{1+y^2+2z^2}}, \frac{x \cdot 4z}{2\sqrt{1+y^2+2z^2}} \right)$$

$$= \boxed{\left( \sqrt{1+y^2+2z^2}, \frac{xy}{\sqrt{1+y^2+2z^2}}, \frac{2xz}{\sqrt{1+y^2+2z^2}} \right)}$$

$$\text{b)} \quad \text{By the chain rule, } \frac{d}{dt}(f(\vec{r}(t))) = \vec{\nabla}f|_{\vec{r}(t)} \cdot \frac{d\vec{r}}{dt}$$

$$\text{so at time } t=0, \text{ we get } \frac{d}{dt}f(\vec{r}(t))|_{t=0} = \vec{\nabla}f|_{\vec{r}=(1,1,1)} \cdot \frac{d\vec{r}}{dt}|_{t=0}.$$

$$= \left( \sqrt{1+1+2}, \frac{1 \cdot 1}{\sqrt{1+1+2}}, \frac{2 \cdot 1 \cdot 1}{\sqrt{1+1+2}} \right) \cdot (3, 4, 5)$$

$$= (2, \frac{1}{2}, 1) \cdot (3, 4, 5) = 2 \cdot 3 + \frac{1}{2} \cdot 4 + 1 \cdot 5 = 6 + 2 + 5 = \boxed{13}.$$

c)  We know that  $f$  DECREASES most rapidly when we move in a direction  $\vec{v}$  OPPOSITE to  $\vec{\nabla}f|_{P_0}$ . (This is because  $D_{\vec{v}}f|_{P_0} = \vec{\nabla}f|_{P_0} \cdot \vec{v} = |\vec{\nabla}f|_{P_0}| \cdot 1 \cdot \cos \alpha$

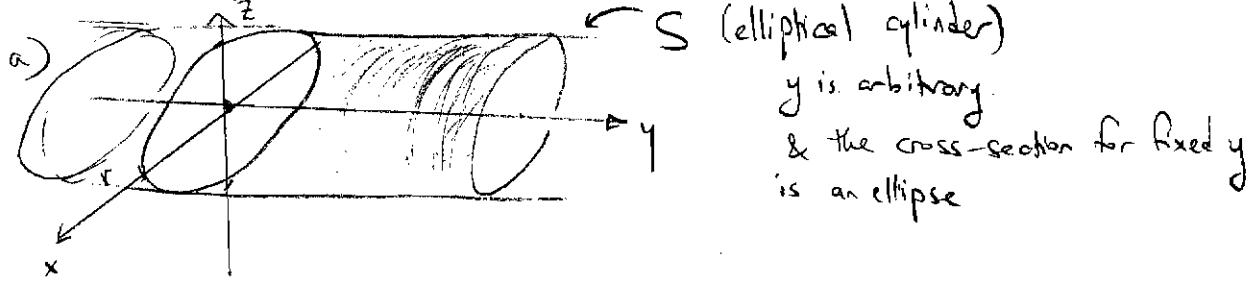
which is most negative when  $\alpha = \pi$  &  $\cos \alpha = -1$ .)

$$\text{Thus } \vec{v} = -\frac{\vec{\nabla}f|_{P_0}}{|\vec{\nabla}f|_{P_0}|} = -\frac{(2, \frac{1}{2}, 1)}{\sqrt{4+\frac{1}{4}+1}} = -\frac{(2, \frac{1}{2}, 1)}{\sqrt{21}/2} = \boxed{-\left(\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)}$$

4. (14 pts total) Consider the function  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  and its level set  $S$ , given by

$$f(x, y, z) = x^2 + 4z^2, \quad S = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + 4z^2 = 8\}.$$

- a) (3 pts) Sketch the surface  $S$ .
- b) (6 pts) Find the equation of the tangent plane to  $S$  at the point  $P_0(2, -1, 1)$ .
- c) (5 pts) Starting from the point  $P_0$  above, we move 0.1 units in the direction of the vector  $\vec{v} = (3, 4, 0)$ . Approximately how much is the resulting change in  $f$ ?



- b)  $S$  is a level set  $\{P \mid f(P) = 8\}$ , and the tangent plane to  $S$  at  $P_0 \in S$  is perpendicular to  $\vec{\nabla}f|_{P_0} = (\partial_x f, \partial_y f, \partial_z f)|_{P_0(2, -1, 1)} = (4, 0, 8)$

$$\begin{aligned} P \in \text{tan plane} &\Leftrightarrow \vec{\nabla}f|_{P_0} \cdot \vec{P}_0 P = 0 \\ &\Leftrightarrow (4, 0, 8) \cdot (x-2, y+1, z-1) = 0 \\ &\Leftrightarrow 4(x-2) + 0 \cdot (y+1) + 8 \cdot (z-1) = 0 \\ &\Leftrightarrow \boxed{4(x-2) + 8(z-1) = 0} \end{aligned}$$

- c) either use  $\Delta f \approx \vec{\nabla}f|_{P_0} \cdot \Delta \vec{r}$  or  $\Delta f \approx (D_{\vec{v}} f|_{P_0}) \Delta s$

in both cases,  $\Delta \vec{r} = (\Delta s) \vec{v}$  where  $\vec{v} = \frac{\vec{\nabla}f}{\|\vec{\nabla}f\|} = \frac{(3, 4, 0)}{\sqrt{9+16+0}} = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$  is the UNIT vector in the direction of  $\vec{\nabla}f$ .

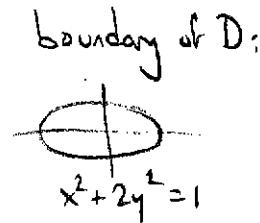
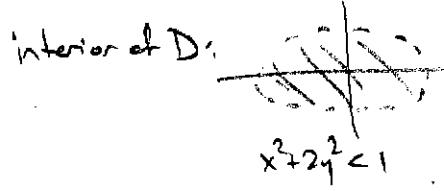
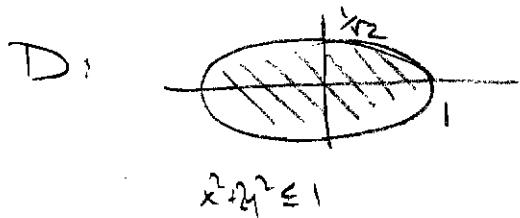
Method 1  $\Delta \vec{r} = (\Delta s) \vec{v} = (0.1) \left(\frac{3}{5}, \frac{4}{5}, 0\right) = \left(\frac{0.3}{5}, \frac{0.4}{5}, 0\right)$  (not worth simplifying yet)

$$\Delta f \approx (4, 0, 8) \cdot \left(\frac{0.3}{5}, \frac{0.4}{5}, 0\right) = 4 \cdot \frac{0.3}{5} + 0 + 0 = \frac{1.2}{5} = \boxed{\frac{6}{25}} \text{ (or } \frac{12}{50} \text{ or } 0.24\text{)}$$

Method 2  $D_{\vec{v}} f|_{P_0} = \vec{\nabla}f|_{P_0} \cdot \vec{v} = (4, 0, 8) \cdot \left(\frac{3}{5}, \frac{4}{5}, 0\right) = \frac{12}{5}$

$$\Delta f \approx (D_{\vec{v}} f|_{P_0}) \Delta s = \left(\frac{12}{5}\right)(0.1) = \text{the same value as above.}$$

5. (12 pts) Find the maximum and minimum values of the function  $f(x, y) = xy$  on the elliptical disk  $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + 2y^2 \leq 1\}$ .



Note  $D$  is closed and bounded, and  $f$  is continuous, so  $f$  attains a max. & min. on  $D$ .

Step 1 check for critical points in the interior of  $D$

$$\nabla f = (y, x) \quad \nabla f = \vec{0} \Leftrightarrow \begin{cases} y=0 \\ x=0 \end{cases} \Leftrightarrow P = P_0(0, 0) \text{ is the only critical point} \\ \text{and } P_0 \in D, \text{ so we must consider it.}$$

Step 2 check the boundary points using Lagrange multipliers

max/min  $f(x, y) = xy$  subject to the constraint  $g(x, y) = x^2 + 2y^2 = 1$ .

Necessary conditions for this are  $\begin{cases} \nabla f|_P = \lambda \nabla g|_P \Leftrightarrow \begin{cases} y = \lambda x \\ x = \lambda y \end{cases} \Rightarrow \begin{cases} y = 2\lambda x \\ x = 4\lambda y \end{cases} \Rightarrow \begin{cases} 2\lambda^2 = 1 \\ x^2 = 4\lambda^2 y^2 \end{cases} \Rightarrow \begin{cases} \lambda^2 = \frac{1}{2} \\ x^2 = 2y^2 \end{cases} \Rightarrow \begin{cases} \lambda = \pm \frac{1}{\sqrt{2}} \\ x^2 = 2y^2 \end{cases} \end{cases} \Rightarrow \begin{cases} x^2 = 2y^2 = \frac{1}{2} \\ x^2 + 2y^2 = 1 \end{cases}$

$$\Leftrightarrow \begin{cases} y = 2\lambda x \xrightarrow{\lambda = \pm \frac{1}{\sqrt{2}}} 2y^2 = 4\lambda^2 xy \\ x = 4\lambda y \xrightarrow{\lambda = \pm \frac{1}{\sqrt{2}}} x^2 = 4\lambda^2 y^2 \end{cases} \Leftrightarrow \begin{cases} 2y^2 = x^2 \\ x^2 + 2y^2 = 1 \end{cases} \text{ (after eliminating } \lambda\text{)}$$

$$\Leftrightarrow x^2 = 2y^2 = \frac{1}{2} \Leftrightarrow x \in \{\pm \frac{1}{\sqrt{2}}\}, y \in \{\pm \frac{1}{2}\} \quad \text{Thus we get 4 candidate}$$

chosen independently from  $x$ .

points on the boundary:

$$\begin{aligned} P_1 & (\frac{1}{\sqrt{2}}, \frac{1}{2}) \\ P_2 & (\frac{1}{\sqrt{2}}, -\frac{1}{2}) \\ P_3 & (-\frac{1}{\sqrt{2}}, \frac{1}{2}) \\ P_4 & (-\frac{1}{\sqrt{2}}, -\frac{1}{2}) \end{aligned}$$

### Table of values

Candidate point  $P(x, y)$

$f(P) = xy$

$$P_0(0, 0)$$

$$0$$

$$P_1(\frac{1}{\sqrt{2}}, \frac{1}{2})$$

$$\frac{1}{2\sqrt{2}}$$

where

$$P_2(\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

$$-\frac{1}{2\sqrt{2}}$$

$-\frac{1}{2\sqrt{2}}$  is the smallest value

$$P_3(-\frac{1}{\sqrt{2}}, \frac{1}{2})$$

$$-\frac{1}{2\sqrt{2}}$$

$+\frac{1}{2\sqrt{2}}$  is the largest value

$$P_4(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

$$\frac{1}{2\sqrt{2}}$$

so the maximum value of  $f$  on  $D$  is  $+\frac{1}{2\sqrt{2}}$ , & it is attained at  $P_1(\frac{1}{\sqrt{2}}, \frac{1}{2})$

the minimum value of  $f$  on  $D$  is  $-\frac{1}{2\sqrt{2}}$ , & it is attained at  $P_2(\frac{1}{\sqrt{2}}, -\frac{1}{2})$  and  $P_3(-\frac{1}{\sqrt{2}}, \frac{1}{2})$

6. (14 pts total) Suggestion for both parts: use Taylor series and  $O(\cdot)$  notation.

a) (7 pts) Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{\ln(1+xy)}{x^2+y^2}$  does NOT exist.

b) (7 pts) Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{\ln(1+x^2y^2)}{x^2+y^2}$  DOES exist.

For both parts:  $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots$   $\ln u = u + O(u^2)$  a finer approximation  
and  $\ln u = O(u)$  a coarser approximation.  
But remember:  $\ln(1+u) \approx u$  for  $u$  small.

a) Scratch  $\ln(1+xy) \approx xy - 2 \frac{xy}{x^2+y^2}$  does not have a limit as  $(x,y) \rightarrow 0$   
by 2 path test along

curves  $P(t,at)$  for a constant  $a$ .  
Actually the curves  $(t,0)$  &  $(t,t)$  distinguish what's happening.

Solution consider  $P(t) = (t,0)$  and  $Q(t) = (t,t)$ . Both are  
curves that approach  $(0,0)$  as  $t \rightarrow 0$ , BUT

$$\lim_{t \rightarrow 0} f(P(t)) = \lim_{t \rightarrow 0} \frac{\ln(1+0)}{t^2+0^2} = \lim_{t \rightarrow 0} \frac{\ln 1}{t^2} = \lim_{t \rightarrow 0} 0 = 0 \quad \boxed{\text{DIFFERENT}}$$

$$\lim_{t \rightarrow 0} f(Q(t)) = \lim_{t \rightarrow 0} \frac{\ln(1+t^2)}{t^2+t^2} = \lim_{t \rightarrow 0} \frac{t^2+O(t^4)}{2t^2} = \lim_{t \rightarrow 0} \frac{1+O(t^2)}{2} = \boxed{\frac{1}{2}}$$

since the limits along the two paths are different,  $\lim_{P \rightarrow (0,0)} f(P)$  DOES NOT EXIST  
by the 2-path test.

use  $u=xy$  & the finer approx

[Remark: you can also use  $\frac{\ln(1+xy)}{x^2+y^2} \leq \frac{xy+O(xy^2)}{x^2+y^2}$  & continue from there.]

As we shall see in part (b),  $\frac{O(xy^2)}{x^2+y^2} \rightarrow 0$  as  $(x,y) \rightarrow (0,0)$ .

b) here  $\frac{\ln(1+x^2y^2)}{x^2+y^2} = \frac{O(x^2y^2)}{x^2+y^2}$  if we use  $u=x^2y^2$  & the coarser approx,  
(as  $(x,y) \rightarrow (0,0)$ ,  $u \rightarrow 0$ )

There are many ways to show that the above  $\rightarrow 0$  as  $(x,y) \rightarrow (0,0)$ .

1st way  $\Delta r = (\Delta x, \Delta y) = (x-\alpha, y-\beta) = (x,y)$  so  $(\Delta s)^2 = x^2+y^2$

also  $x^2 = (\Delta x)^2 \leq (\Delta s)^2$  &  $y^2 \leq (\Delta s)^2$  similarly, so  $x^2y^2 \leq (\Delta s)^4$   
i.e.  $x^2y^2 = O((\Delta s)^4)$

Then  $\frac{O(x^2y^2)}{x^2+y^2} = \frac{O((\Delta s)^4)}{(\Delta s)^2} = O((\Delta s)^2) \rightarrow 0$  as  $\Delta s \rightarrow 0$ .

2nd way use  $0 \leq \frac{x^2}{x^2+y^2} \leq 1$  to get  $0 \leq \frac{x^2y^2}{x^2+y^2} \leq y^2$  & use  $\frac{0 \rightarrow 0}{y^2 \rightarrow 0} \Leftrightarrow P \rightarrow (0,0)$   
+ the sandwich theorem.

3rd way in polar coordinates, we have  $\frac{O(r^4 \cos^2 \theta \sin^2 \theta)}{r^2} = O(r^2 \cos^2 \theta \sin^2 \theta) = O(r^2) \rightarrow 0$   
BECAUSE  $|6^2 \cos^2 \theta \sin^2 \theta| \leq 1$  i.e.  $r^2$  is bounded regardless of  $\theta$ .